Proceedings of the 24th UK Conference of the
Association for Computational Mechanics in Engineering
31 March - 01 April 2016, Cardiff University, Cardiff

Two dimensional configurational-force-driven crack propagation using the discontinuous Galerkin method with rp-adaptivity

***Robert Bird[1], William M. Coombs[1] and Stefano Giani[1]**

[1]School of Engineering and Computing Sciences, Durham University, South Road, Durham, DH1 3LE

*robert.e.bird@durham.ac.uk

ABSTRACT

This paper presents a quasi-static configurational force (CF) brittle fracture propagation method, [1], using the discontinuous Galerkin (dG) symmetric interior penalty (SIP) method, [2]. The method is derived from the first law of thermodynamics with consideration of the Griffith fracture criterion [1]. The criterion is evaluated by finding the difference between the power applied to the domain and the rate of internal energy change at every point in the domain. If a node within the element mesh satisfies the criterion, a crack will propagate in the CF direction. Around the crack tip the advantage of element specific degrees of freedom in dG methods enables simple p-adaptivity to determine the CF in the spatial domain. In the material domain r-adaptivity is implemented, where the CF direction is used to align element edges, which are then split to propagate the crack.

Key Words: *Crack propagation; configurational force; discontinuous Galerkin; symmetric interior penalty; rp-adaptation*

1. Introduction

Fracture propagation is the generation of new surfaces in a domain through crack growth. Numerical implementations of brittle fracture propagation is relatively rare and remains to be one of the most significant challenges in solid mechanics. Numerical frameworks must be able to predict the initiation of a crack and the subsequent path. A promising technique, known as r-adaptivity, which mitigates influence from the mesh on the direction of the crack path has been presented by Miehe *et al.*, [1, 3, 4]. The method is based on the concept of material configuration force, [5, 6, 7], with the use of a Griffith brittle fracture failure criterion to determine crack growth. The unique aspect of the method is the realignment of an element face with the direction of the configurational force vector at the crack tip. The face is then split to propagate the crack.

In this paper the SIP dG finite element (FE) scheme, [8], is used to apply the r-adaptivity method in conjunction with hierarchical shape functions for p-adaptivity [9]. The connectivity between elements in the dG space is generated through dG face stiffness terms, not shared degrees of freedom. The implication is that new surfaces can be generated by removing dG face terms between elements, in the global element stiffness matrix. The second implication is adjacent elements can have varying degrees of freedom without concern over hanging nodes. Moreover, it is simple to incorporate new higher order elements into an already existing data structure as it is only necessary to ensure the new degrees of freedom are unique to a single element and that new dG face stiffness terms are calculated for the higher order element's connectivity to its adjacent elements.

In this paper the SIP dG method is initially defined as the scheme to model the CF crack growth. This is then followed by a definition of the CF, the growth rate and the corresponding consistency conditions. Next the method for applying rp-adaptivity within a SIP dG framework is outlined. Lastly results are represented to validate CF values and the crack propagation path against analytical solutions.

2. Discontinuous Galerkin method for linear elasticity

Here we introduce SIP dG for the linear elasticity problem,

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in} \quad \Omega, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{g}_N \quad \text{on} \quad \partial\Omega_N \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g}_D \quad \text{on} \quad \partial\Omega_D. \tag{1}$$

Ω is a polygonal domain and $\boldsymbol{f}$ is a force vector in $[\boldsymbol{L}^2(\Omega)]^2$. Ω is divided into elements K. We set $W_h = \{\boldsymbol{w} \in [L^2(\Omega)]^2 : \forall K, \boldsymbol{w}|_K \in \mathscr{P}_{p_K}(K)\}$ where $\mathscr{P}_{p_K}(K)$ is the space of polynomial functions of degree at most $p_K \geqslant 1$ on K. The elementwise approximation of $\boldsymbol{u}$ is defined $\boldsymbol{u_h} \in W_h$. (1) is weakly satisfied by $\boldsymbol{u_h}$ such that $\boldsymbol{a}(\boldsymbol{u_h}, \boldsymbol{v}) = \boldsymbol{l}(\boldsymbol{v})$. Here, $\boldsymbol{a}(\boldsymbol{u_h}, \boldsymbol{v}) = \sum_{K\in\Omega} \boldsymbol{a}_K(\boldsymbol{u_h}, \boldsymbol{v})$ and $\boldsymbol{l}(\boldsymbol{v}) = \sum_{K\in\Omega} \boldsymbol{l}_K(\boldsymbol{v})$ where,

$$\boldsymbol{a}_K(\boldsymbol{u_h}, \boldsymbol{v}) = (\boldsymbol{\sigma}(\boldsymbol{u_h}), \boldsymbol{\epsilon}(\boldsymbol{v}))_K - \langle\{\boldsymbol{\sigma}(\boldsymbol{u_h})\}, [\![\boldsymbol{v}]\!]\rangle_{\partial K\setminus\partial\Omega} - \langle[\![\boldsymbol{u_h}]\!], \{\boldsymbol{\sigma}(\boldsymbol{v})\}\rangle_{\partial k\setminus\partial\Omega} + \beta\langle h^{-1}[\![\boldsymbol{u_h}]\!], [\![\boldsymbol{v}]\!]\rangle_{\partial K\setminus\partial\Omega}, \tag{2}$$

and

$$\boldsymbol{l}_K(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_K. \tag{3}$$

β is a penalty term for linear elastic SIP dG defined by [10], h is the element face length, and $\{\cdot\}$, $[\![\cdot]\!]$, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are defined in [2]. The displacement boundary conditions are applied in the strong form.

3. Evaluation of the configurational force for fracture mechanics

Miehe *et al.*, [1], provide a robust derivation based on the work on the configuration force by Eshelby [5, 6, 11], to determine the configurational force as,

$$\boldsymbol{G_I} = \lim_{|C|\to 0} \int_C \boldsymbol{\Sigma}\cdot\boldsymbol{n}\mathrm{d}S, \tag{4}$$

where $\boldsymbol{\sigma} = \partial_{\boldsymbol{\epsilon}}\hat{\psi}(\boldsymbol{\epsilon})$ and $\boldsymbol{\Sigma} = \hat{\psi}(\boldsymbol{\epsilon})\mathbf{1} - \boldsymbol{h_d}^\top\boldsymbol{\sigma}$ are the symmetric Cauchy stress tensor and non-symmetric Eshelby stress tensor, $\boldsymbol{n}$ is an inward normal to the crack, $\hat{\psi}(\boldsymbol{\epsilon})$ is the free energy function, C is a surface around the crack tip and $\boldsymbol{h_d}$ is the displacement gradient. The method is cast within an arbitrary Lagrangian-Eulerian description of motion requiring a reference material domain and spacial domain. The velocity of the crack, $\Delta\boldsymbol{a_I}$, is controlled by the isotropic Griffith failure criterion function $\hat{\phi}(\boldsymbol{G_I}) = |\boldsymbol{G_I}| - g_c \leq 0$ where g_c is the material parameter specifying the critical energy release pre unit length. Propagation of the crack tip is controlled by the following consistency conditions,

$$\Delta\boldsymbol{a_I} = \Delta\gamma_I \frac{\boldsymbol{G}_I}{|\boldsymbol{G}_I|}, \quad \text{where} \quad \Delta\gamma_I \geq 0, \quad \hat{\phi}(\boldsymbol{G_I}) \leq 0, \quad \text{and} \quad \Delta\gamma_I\hat{\phi}(\boldsymbol{G_I}) = 0. \tag{5}$$

If the Griffith failure criterion is satisfied then $\Delta\gamma_I = H_c$, where H_c is the length of the face to be split.

4. RP-adaptivity

For our simulations hierarchical shape functions are employed for both the material and spacial domain. All elements are triangular and have a shape function order of 1 except those at the crack tip with an order p. The configurational force is evaluated at the crack tip using the domain method presented by Denzer *et al.*, [12]. If $|\boldsymbol{G}_I| \geq g_c$ is true then the crack will propagate in the direction $\boldsymbol{G}_I/|\boldsymbol{G}_I|$ and the rp-adaptivity method will be applied, see Figure 1.

The r-adaptivity step occurs in two stages. First, the face in the material domain most aligned to the CF direction is rotated about the crack node to be exactly parallel with the CF direction, [1]. Secondly, the crack surface is propagated by removing dG face stiffness terms from all future stiffness calculations for the manoeuvred face.

To initiate p-adaptivity all components of (2) for elements 1-6, and all external elements sharing faces with elements 1-6, are recalculated and steered back into the global stiffness matrix. This is necessary to accommodate changes in element topology, crack surfaces and element order. Management of the data structure is relatively simple, the only consideration being degrees of freedom must be unique to a single element.

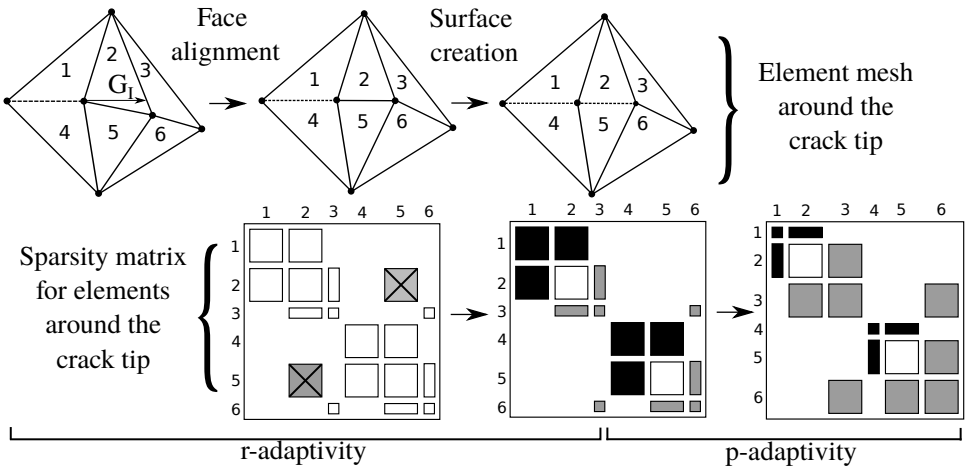
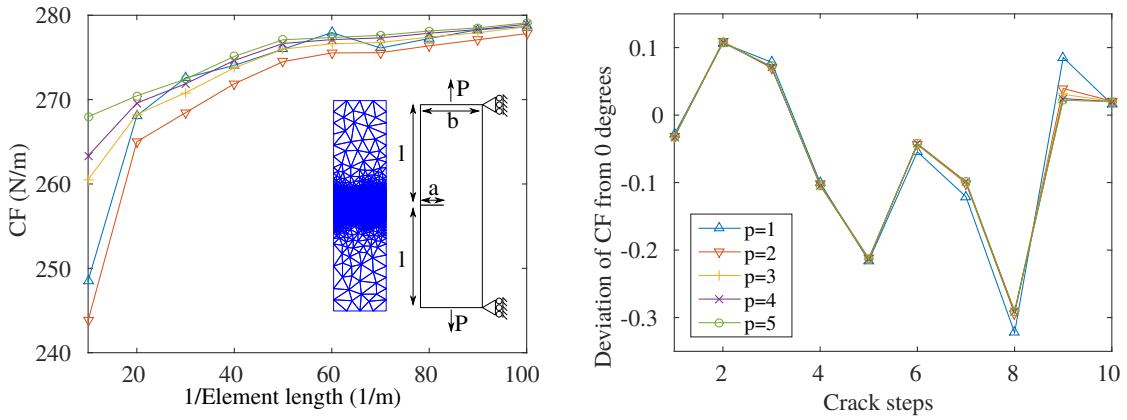


Figure 1: The rp-adaptivity for the element mesh at the crack tip with its corresponding sparsity matrix. Sparsity components with a grey box and X are removed dG face stiffness terms to increase the crack length. Black components are elements which undergo a reduction in polynomial order, white no change, and grey an increase.

5. Experimental results



(a) CF value for varying element length and local p-adaptivity at crack tip.

(b) CF angle deviation from 0 degrees for a developing crack.

Figure 2: **a)** CF values for a force applied to a static crack, inset the experimental setup and mesh. **b)** CF angle for a propagating crack with a displacement boundary condition. CF values were calculated using the method presented by Denzer *et al.* [12], with a radius of 0.05 m.

A plane stress experiment using the SIP dG method was performed to validate CF values at the crack tip of a stationary crack against a known solution provided by [13]. The experimental setup is described in Figure 2a, the geometric parameters a b and l have values 0.1 m, 0.5 m and 1 m respectively. The plate has a Young's modulus of 208 GPa and Poisson's ratio of 0.3. The tensile stress, P, was 10 Mpa.

The unstructured mesh consists of triangular element generated using Triangle, [14]. The element face length is varied around the crack tip. The element polynomial order of elements that share a node at the crack tip is defined p, all other elements have an order 1. The CF force values for varying element length and polynomial order is displayed in Figure 2a.

In comparison to the analytical solution of 256.65 N/m, the results converge to a value with an error $\approx 8\%$. The error value similar to those expressed by [1] for the same experiment. Faster convergence is

achieved for meshes with higher element order around the crack tip.

A second experiment was performed to test how varying polynomial order of elements on the advancing crack tip effected the validity of the CF angle against the expected value of 0^o, as shown in Figure 2b. The problem was set up with the same geometric and material parameters as before, however an axial load in the form of a 0.1 m displacement boundary condition on either end of the plate is applied instead of P. All elements other than those at the crack tip have a polynomial order 1. The element length at the crack tip was 0.01m.

6. Conclusion

This paper has presented an rp-adaptivity method to model small strain quasi-static configurational-force-driven crack propagation. Problems are restricted to two-dimensions with linear geometric and material properties. A comparison of the CF values between analytical solutions and those calculated from surface integrals is presented. The application and results of applying the SIP dG method with hierarchical shape functions for a propagating and stationary crack have been presented.

References

[1] C. Miehe and E. Gürses, "A robust algorithm for configurational-force-driven brittle crack propagation with r-adaptive mesh alignment," *INT J NUMER METH ENG*, vol. 72, no. 2, pp. 127–155, 2007.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, "Unified analysis of discontinuous Galerkin methods for elliptic problems," *SIAM J NUMER ANAL*, vol. 39, no. 5, pp. 1749–1779, 2002.

[3] C. Miehe, E. Gürses, and M. Birkle, "A computational framework of configurational-force-driven brittle fracture based on incremental energy minimization," *INT J FRACTURE*, vol. 145, no. 4, pp. 245–259, 2007.

[4] E. Gürses and C. Miehe, "A computational framework of three-dimensional configurational-force-driven brittle crack propagation," *COMPUT METHOD APPL M*, vol. 198, no. 15, pp. 1413–1428, 2009.

[5] J. D. Eshelby, "The force on an elastic singularity," *PHILOS T ROY SOC A*, vol. 244, no. 877, pp. 87–112, 1951.

[6] J. D. Eshelby, "Energy relations and the energy-momentum tensor in continuum mechanics," in *Fundamental Contributions to the Continuum Theory of Evolving Phase Interfaces in Solids*, pp. 82–119, Springer, 1999.

[7] J. R. Rice, "A path independent integral and the approximate analysis of strain concentration by notches and cracks," *J APPL MECH*, vol. 35, no. 2, pp. 379–386, 1968.

[8] J. Douglas and T. Dupont, "Interior penalty procedures for elliptic and parabolic Galerkin methods," in *Lecture notes in Phys. 58*, Springer-Verlag, 1976.

[9] P. Solin, K. Segeth, and I. Dolezel, *Higher-order finite element methods*. CRC Press, 2003.

[10] P. Hansbo and M. G. Larson, "Energy norm a posteriori error estimates for discontinuous Galerkin approximations of the linear elasticity problem," *COMPUT METHOD APPL M*, vol. 200, no. 45, pp. 3026–3030, 2011.

[11] J. D. Eshelby, "The determination of the elastic field of an ellipsoidal inclusion, and related problems," in *PHILOS T ROY SOC A*, vol. 241, pp. 376–396, The Royal Society, 1957.

[12] R. Denzer, F. J. Barth, and P. Steinmann, "Studies in elastic fracture mechanics based on the material force method," *INT J NUMER ANAL MET*, vol. 58, no. 12, pp. 1817–1835, 2003.

[13] D. P. Rooke and D. J. Cartwright, "Compendium of stress intensity factors," *Procurement Executive, Ministry of Defence. H. M. S. O. 1976, 330 p(Book).*, 1976.

[14] J. R. Shewchuk, "Triangle: Engineering a 2D Quality Mesh Generator and Delaunay Triangulator," in *Applied Computational Geometry: Towards Geometric Engineering* (M. C. Lin and D. Manocha, eds.), vol. 1148 of *Lecture Notes in Computer Science*, pp. 203–222, Springer-Verlag, May 1996. From the First ACM Workshop on Applied Computational Geometry.